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Bounds on the 2-Rainbow Domination Number of Graphs

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Abstract A 2-*rainbow domination function* of a graph G is a function f that assigns to each vertex a set of colors chosen from the set $\{1, 2\}$, such that for any $v \in V(G)$, $f(v) = \emptyset$ implies $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$. The 2-*rainbow domination number* $\gamma_{r2}(G)$ of a graph G is the minimum $w(f) = \Sigma_{v \in V}|f(v)|$ over all such functions f. Let G be a connected graph of order $|V(G)| = n \geq 3$. We prove that $\gamma_{r2}(G) \leq 3n/4$ and we characterize the graphs achieving equality. We also prove a lower bound for 2-rainbow domination number of a tree using its domination number. Some other lower and upper bounds of $\gamma_{r2}(G)$ in terms of diameter are also given.

Keywords Domination number · 2-Rainbow domination number · Cartesian product

Mathematics Subject Classification 05C69

1 Introduction

We follow the notation of [1] in this paper. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. P_k and C_k denote a path and a cycle of order k, respectively. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set

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$S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. The *diameter* of G is the maximum distance between vertices of G, denoted by $diam(G)$. A *penultimate vertex* is any neighbor of a vertex with degree one (the vertex of degree one is also called a *leaf* in a tree), and a *pendent* edge is an edge incident with a vertex of degree one. A *star* is a tree isomorphic to a bipartite graph $K_{1,k}$ for $k \geq 1$. A *double-star* $DS_{r,s}$ is a tree with diameter 3 in which there are exactly two penultimate vertices with degrees $r+1$ and $s+1$, respectively. A set $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A thorough study of domination concepts appears in [7]. For a pair of graphs G and H, the *Cartesian product* $G \Box H$ of G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other.

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, \ldots, k\}$; that is, $f : V(G) \to \mathscr{P}(\{1, \ldots, k\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$, we have

$$\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\},$$

then f is called a *k-rainbow dominating function* (kRDF) of G. The weight, $w(f)$, of a function f is defined as $w(f) = \Sigma_{v \in V(G)} |f(v)|$. The minimum weight of a k-rainbow dominating function is called the *k-rainbow domination number* of G, which we denote by $\gamma_{rk}(G)$. We say that a function f is a $\gamma_{rk}(G)$-*function* if it is a kRDF and $w(f) = \gamma_{rk}(G)$. The concept of rainbow domination was introduced in [3], and used in obtaining some bounds on the paired-domination number of Cartesian products of graphs, see also [2]. A more ambitious motivation for the introduction of this invariant was inspired by the following famous open problem [9]:

Vizing's Conjecture. For any graphs G and H, $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.

In the language of domination of Cartesian products, Hartnell and Rall [6] obtained a couple of observations about rainbow domination, for instance, $\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_{rk}(G) \leq k\gamma(G)$. Rainbow domination of a graph G coincides with the ordinary domination of the Cartesian product of G with the complete graph, in particular $\gamma_{r2}(G) = \gamma(G \Box K_2)$ for any graph G [3]. Notably a lower bound for the 2-rainbow domination number of a graph expressed in terms of its ordinary domination could bring a new approach to the much desired proof of Vizing's conjecture. In particular, Brešar et al. [3] proposed the following problem:

Problem 1 (Brešar et al. [3]). *For any graphs G and H, $\gamma_{r2}(G \Box H) \geq \gamma(G)\gamma(H)$.*

Nevertheless the concept of rainbow domination seems to be of independent interest as well and it attracted several authors who provided structural and algorithmic results on this invariant [4,5,8,10,11]. In particular, it was shown that the problem of deciding if a graph has a 2-rainbow dominating function of a given weight is NP-complete even when restricted to bipartite graphs or chordal graphs [4]. Also a few exact values and bounds for the 2-rainbow domination number were given for some special classes of graphs, including generalized Petersen graphs [4,11].

For a graph G, let $f : V(G) \to \mathscr{P}(\{1, 2\})$ be a 2RDF of G and (V_0, V_1^1, V_1^2, V_2) be the ordered partition of $V(G)$ induced by f, where $V_0 = \{v \in V(G) \mid f(v) = \emptyset\}$, $V_1^1 = \{v \in V(G) \mid f(v) = \{1\}\}$, $V_1^2 = \{v \in V(G) \mid f(v) = \{2\}\}$ and $V_2 = \{v \in V(G) \mid f(v) = \{1, 2\}\}$. Note that there exists a 1-1 correspondence between the functions $f : V(G) \to \mathscr{P}(\{1, 2\})$ and the ordered partitions (V_0, V_1^1, V_1^2, V_2) of $V(G)$. Thus we will write $f = (V_0, V_1^1, V_1^2, V_2)$ for simplicity.

In this paper we present some general bounds on the 2-rainbow domination number of a graph that are expressed in terms of the order and domination number of a graph. More specifically, we show that $\gamma_{r2}(G) \leq 3|V(G)|/4$ and we characterize the graphs achieving equality. We also prove a lower bound for the 2-rainbow domination number of a tree using its domination number. The latter lower bound goes in the direction of the original goal, mentioned above, to obtain a new approach for establishing Vizing's conjecture. Some other lower and upper bounds of $\gamma_{r2}(G)$ in terms of diameter are also given.

2 Main Results

Our aim in this section is to determine some bounds on the 2-rainbow domination number of graphs.

2.1 Upper Bounds

We first recall a few definitions. A *subdivision* of an edge uv is obtained by removing edge uv, adding a new vertex w, and adding edges uw and vw. Let $t \geq 2$. A *spider* (*wounded spider*) is the graph formed by subdividing some edges (at most $t-1$ edges) of a star $K_{1,t}$. The unique center of $K_{1,t}$ is also called the *center* of the spider. Only one vertex of the spider P_4 can be called the center.

Proposition 1 *Let G be a spider of order $|V(G)| = n \geq 3$, then $\gamma_{r2}(G) \leq 3n/4$. Moreover, the equality only holds for a path of order four.*

Proof Let u be the center of G. Suppose u has x penultimate neighbors and y non-penultimate neighbors. Then $n = 2x + y + 1$.

If $x \geq 3$ or $y \geq 2$, we set

$$f(v) = \begin{cases} \{1, 2\} & \text{v=u}, \\ \{1\}\text{or } \{2\} & \text{d(u,v)=2}, \\ \emptyset & \text{otherwise}. \end{cases}$$

If $x = 2$ and $y \leq 1$, we set

$$f(v) = \begin{cases} \{1\} & \text{v=u}, \\ \{2\} & \text{v is a leaf}, \\ \emptyset & \text{otherwise}. \end{cases}$$

In both cases, $\gamma_{r2}(G) \le w(f) < 3n/4$.

If $x = y = 1$, then G is a path of order four. Clearly, $\gamma_{r2}(G) = 3 = 3n/4$.

Theorem 1 *Let T be a tree of order $n \ge 3$, then $\gamma_{r2}(T) \le 3n/4$.*

Proof We use induction on n. The base step handles trees with few vertices or small diameter. If $diam(T) = 2$, then T has a dominating vertex, and $\gamma_{r2}(T) \le 2$. This beats $n \ge 3$. If $diam(T) = 3$, then T has a dominating set of size two, which yields $\gamma_{r2}(T) \le 4$. This handles the desired bound for such trees with at least six vertices. When $n = 4$ or $n = 5$, then T is a spider and the theorem holds by Proposition 1. Moreover, if T is a path of order four, then it achieves this bound.

Hence we may assume that $diam(T) \ge 4$. Given a subtree T' with n' vertices, where $n' \ge 3$, the induction hypothesis yields a 2RDF f' of T' with weight at most $3n'/4$. We find such T' and add a bit more weight to obtain a 2RDF f of T. Let P be a longest path in T chosen to maximize the degree of the penultimate vertex v on it, and let u be the non-leaf neighbor of v.

Case 1 $d_T(v) > 2$.
We obtain T' by deleting v and its leaf neighbors. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = \{1, 2\}$ and $f(x) = \emptyset$ for each leaf x adjacent to v. Since color set $\{1, 2\}$ on v takes care of its neighbors, f is a 2RDF for T. Since $diam(T) \ge 4$, we have $n' \ge 3$, and $w(f) = w(f') + 2 \le 3n'/4 + 2 \le 3(n - 3)/4 + 2 < 3n/4$.

Case 2 $d_T(v) = d_T(u) = 2$.
We obtain T' by deleting u and v and the leaf neighbor l of v. If $n' = 2$, then T is a path of order five and has a 2RDF of weight $3 < 3n/4$. Otherwise, the induction hypothesis applies. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = \{1, 2\}$ and $f(u) = f(l) = \emptyset$. Again f is a 2RDF, and the computation $w(f) < 3n/4$ is the same as in Case 1.

Case 3 $d_T(v) = 2$ and $d_T(u) > 2$.
By the choice of path P, every penultimate neighbor of u has degree 2.

Subcase 3.1. Every neighbor of u is penultimate or a leaf.
Then $diam(T) = 4$ and T is a spider. By Proposition 1, $\gamma_{r2}(T) < 3n/4$, since T is not a path of order four.

Subcase 3.2. There exists a neighbor t of u which is neither penultimate nor a leaf.

Then $T - tu$ contains two components T' and T'' such that T'' is a spider containing u. Now $|V(T')| = n' \ge 3$ and the induction hypothesis applies that $\gamma_{r2}(T') \le 3|V(T')|/4 = 3n'/4$. By Proposition 1, $\gamma_{r2}(T'') \le 3|V(T'')|/4$. Hence $\gamma_{r2}(T) \le \gamma_{r2}(T') + \gamma_{r2}(T'') \le 3n/4$.

Let L_k consist of the disjoint union of k copies of P_4 plus a path of order k through the center vertices of these copies, as illustrated in Fig. 1. Let G be a graph having an induced subgraph P_4 such that only the center of P_4 can be adjacent to the vertices in $G - P_4$, then every 2RDF of G must have weight at least 3 on P_4. In L_k, there are k disjoint P_4 of this form, so $\gamma_{r2}(L_k) \ge 3k = 3n/4$. Indeed, we can assemble such copies of P_4 in many ways, and this allows us to characterize the trees achieving equality in Theorem 1.

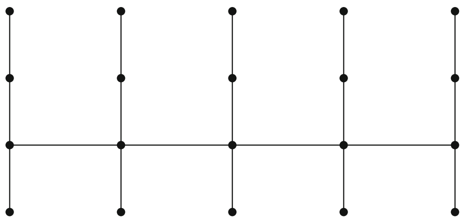

Fig. 1 The tree L_5

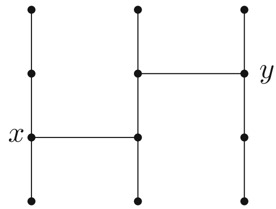

Fig. 2 A spanning subgraph H' of T

Theorem 2 *Let T be a tree of order $n \geq 3$. Then $\gamma_{r2}(T) = 3n/4$ if and only if $V(T)$ can be partitioned into sets inducing P_4 such that the subgraph induced by the center vertices of these P_4 is connected.*

Proof We have observed that if an induced subgraph H of G is isomorphic to P_4, and its noncenter vertices have no neighbors outside H in G, then every 2RDF of G must have weight at least 3 on $V(H)$. Thus in any tree with the structure described, weight at least 3 is needed on every P_4 in the specified partition. To show that equality requires this structure, we examine the cases more closely in the proof of Theorem 1. The proof is by induction on n. In the base cases and Cases 1 and 2, we produce a 2RDF with weight less than $3n/4$ except for P_4. Define u, T', T'', n', t as in the inductive part of Case 3. The equality holds only if $n' = n - 4$ and T'' is a P_4 path. Equality also requires $\gamma_{r2}(T') = 3n'/4$, so by the induction hypothesis T' has the specified form.

Next we show no copy of P_4 in T such that both the two penultimate vertices on P_4 with degree at least three in T. Suppose there is a spanning subgraph H' isomorphic to the graph shown in Fig. 2, then we give a 2RDF f for H' as follows:

$$f(v) = \begin{cases} \{1, 2\} & \text{v=x or y,} \\ \{1\} & \text{v} \notin \text{N[x]} \cup \text{N[y]}, \\ \emptyset & \text{otherwise.} \end{cases}$$

By Theorem 1, $\gamma_{r2}(T) \leq \gamma_{r2}(H') + \gamma_{r2}(T - H') \leq 8 + 3(n - 12)/4 < 3n/4$, a contradiction.

Recall that the *corona* HoK_1 of a graph H is obtained by attaching one pendent edge at each vertex of H. Since the rainbow domination number does not increase when edges are added to a graph, we infer from Theorems 1 and 2 the following general upper bound.

Corollary 1 *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{r2}(G) \leq 3n/4$. Moreover, the equality holds if and only if G is P_4 or C_4oK_1 or $V(G)$ can be partitioned* *into k copies of* $P_4(k \geq 3)$ *and all the copies of* P_4 *can only be connected by their centers.*

Proof If G has the specified form, then for each copy of P_4 in the partition of $V(G)$, every 2RDF of G puts weight at least 3 on it.

Suppose $\gamma_{r2}(G) = 3n/4$ and G is not a tree. Since adding edges can not increase the 2-rainbow domination number, every spanning tree of G has the form specified in Theorem 2. If $n = 4$, then G is P_4. If $n = 8$, then it is easy to check that the only extremal graph is $C_4 o K_1$. If $n \geq 12$, let T be a spanning tree of G has the form specified in Theorem 2. G is not a tree, so there exists an edge $e \in E(G) - E(T)$ such that $T \cup e$ contains a cycle C. Without loss of generality, assume e is not an edge connecting two centers in T. If C contains no edge joining the centers in T, i.e., C is formed by some vertices of a copy P_4, then a 2RDF with weight $3n/4 - 1$ can be found, since we only need to put weight 2 on the vertices of this copy of P_4(this copy of P_4 is then a cycle or contains a vertex of degree three) to take care of this copy of P_4. If C goes through an edges e' joining the centers of two copies of P_4 in T, then $\gamma_{r2}(T \cup e - e') < 3n/4$ since tree $T \cup e - e'$ is not the form specified in Theorem 2. Hence $\gamma_{r2}(G) < 3n/4$. The proof is complete.

The following result for the 2-rainbow domination number of paths is given by Brešar and Kraner Šumenjak.

Proposition 2 ([4]) $\gamma_{r2}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.

We conclude this subsection with an upper bound in terms of diameter.

Theorem 3 *For any connected graph G on n vertices,*

$$\gamma_{r2}(G) \leq n - \left\lceil \frac{diam(G) - 1}{2} \right\rceil.$$

Furthermore, this bound is sharp.

Proof Let $P = v_1 v_2 \cdots v_{diam(G)+1}$ be a diametral path in G and f be a γ_{r2}-function of P. By Proposition 2, the weight of f is $\left\lfloor \frac{diam(G)+1}{2} \right\rfloor + 1$. Define $g : V(G) \to \mathscr{P}(\{1, 2\})$ by $g(x) = f(x)$ for $x \in V(P)$ and $g(x) = \{1\}$ for $x \in V(G) - V(P)$. Obviously g is a 2RDF for G. Hence,

$$\gamma_{r2}(G) \leq w(f) + (n - diam(G) - 1) = n - \left\lceil \frac{diam(G) - 1}{2} \right\rceil$$

The family of all paths achieves the bound, and the proof is complete.

2.2 Lower Bounds

We present a lower bound on the 2-rainbow domination number of a tree expressed in terms of its domination number, maximum degree, and the number of its leaves and penultimate vertices. Given a tree T, we denote by $\ell(T)$ the number of leaves in T, and by $p(T)$ the number of penultimate vertices in T.

Theorem 4 *For any tree T on at least three vertices, $\gamma_{r2}(T) \geq \gamma(T) + \left\lceil \frac{\ell(T)-p(T)}{\Delta(T)} \right\rceil$, where $\Delta(T)$ denotes the maximum degree in T.*

Proof The proof is by induction on the order of T. First we handle trees with small diameter. If $diam(T) \leq 2$ then $\gamma(T) = 1$, $\gamma_{r2}(T) = 2$, and one can easily find that the required inequality holds. Moreover, we have $\gamma_{r2}(G) = \gamma(T) + \lceil \frac{\ell(T)-p(T)}{\Delta(T)} \rceil$ precisely when T is isomorphic to $K_{1,r}$ for $r > 1$. If $diam(T) = 3$ then another simple analysis shows that the inequality holds, and the equality is achieved for $DS_{r,s}$ with $r \geq s \geq 4$ and $DS_{r,1}$ with $r \geq 2$.

Let T be a tree. By the above we may assume that $diam(T) \geq 4$. Let P be a diametral path with two penultimate vertices, say v and v', of P. Without loss of generality, we assume $d_T(v) \leq d_T(v')$. Let u be the neighbor of v that is not a leaf (hence u also lies on P). Let L denote the vertex set containing v and all leaves adjacent to v and $F(u)$ be all the possible color sets of vertex u among all γ_{r2}-function of $T - L$. Then $\gamma(T - L) \leq \gamma(T) \leq \gamma(T - L) + 1$, $\Delta(T - L) = \Delta(T)$ and $p(T - L) \leq p(T) \leq p(T - L) + 1$.

Case 1. $d_T(v) = 2$ and $F(u) = \{\{1\}, \{2\}, \{1, 2\}\}$.
In this case $\gamma_{r2}(T) = \gamma_{r2}(T-L)+1$. By induction hypothesis $\gamma_{r2}(T-L) \geq \gamma(T - L) + \left\lceil \frac{\ell(T-L)-p(T-L)}{\Delta(T-L)} \right\rceil$. We finally get

$$\begin{aligned}
\gamma_{r2}(T) &= \gamma_{r2}(T - L) + 1 \\
&\geq \gamma(T - L) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil + 1 \\
&\geq \gamma(T) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil \\
&\geq \gamma(T) + \left\lceil \frac{\ell(T) - p(T)}{\Delta(T)} \right\rceil.
\end{aligned}$$

Since if $p(T - L) = p(T)$, then $\ell(T - L) = \ell(T)$. Otherwise $p(T - L) = p(T) - 1$ and $\ell(T - L) = \ell(T) - 1$. The last inequality is obtained.

Case 2. $d_T(v) \geq 3$ or $d_T(v) = 2$ and $F(u) = \{\emptyset\}$.
In this case $\gamma_{r2}(T) = \gamma_{r2}(T - L) + 2$. Then we get

$$\begin{aligned}
\gamma_{r2}(T) &= \gamma_{r2}(T - L) + 2 \\
&\geq \gamma(T - L) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil + 2 \\
&\geq \gamma(T) + \left\lceil \frac{\ell(T - L) - p(T - L)}{\Delta(T - L)} \right\rceil + 1 \\
&\geq \gamma(T) + \left\lceil \frac{\ell(T) - p(T)}{\Delta(T)} \right\rceil.
\end{aligned}$$

In the last inequality we use that the excess of leaves in T with respect to $T - L$ does not go beyond $\Delta(T)$.

In the above proof we mentioned several examples of trees with diameter at most 3 that achieve the bound in Theorem 4. We pose a characterization of all these extremal graphs as an open problem.

Next we give a lower bound of the 2-rainbow domination number of an arbitrary graph in terms of its diameter.

Theorem 5 *For any connected graph G, $\gamma_{r2}(G) \geq \left\lceil \frac{2diam(G)+2}{5} \right\rceil$.*

Proof Let $f = (V_0, V_1^1, V_1^2, V_2)$ be a 2RDF of G. Consider an arbitrary path of length $diam(G)$. This diametral path includes at most two edges from the induced subgraph $\langle N[v] \rangle_G$ for each vertex $v \in V_1^1 \cup V_1^2 \cup V_2$. Furthermore, if vertex $v \in V_0$, then it is adjacent to a vertex with color set $\{1, 2\}$, or adjacent to two different vertices with color set $\{1\}$ and $\{2\}$, respectively. Hence excluding the edges mentioned above, the diametral path includes at most $\min\{|V_1^1|, |V_1^2|\} + |V_2| - 1$ other edges joining the neighborhoods of the vertices of $V_1^1 \cup V_1^2 \cup V_2$. Therefore

$$\begin{aligned} diam(G) &\leq 2(|V_1^1| + |V_1^2| + |V_2|) + \min\{|V_1^1|, |V_1^2|\} + |V_2| - 1 \\ &\leq 2(|V_1^1| + |V_1^2| + |V_2|) + (|V_1^1| + |V_1^2|)/2 + |V_2| - 1 \\ &= 5/2(|V_1^1| + |V_1^2| + 2|V_2|) - 2|V_2| - 1 \\ &\leq 5/2\gamma_{r2}(G) - 1. \end{aligned}$$

Then the desired result follows.

Clearly, the bound of Theorem 5 is sharp, e.g. for G isomorphic to P_3 or C_4.

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