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OUTER INDEPENDENT RAINBOW DOMINATING FUNCTIONS IN GRAPHS

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Abstract. A 2-rainbow dominating function (2-rD function) of a graph $G = (V, E)$ is a function $f : V(G) \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ having the property that if $f(x) = \emptyset$, then $f(N(x)) = \{1, 2\}$. The 2-rainbow domination number $\gamma_{r2}(G)$ is the minimum weight of $\sum_{v \in V(G)} |f(v)|$ taken over all 2-rainbow dominating functions f. An outer-independent 2-rainbow dominating function (OI2-rD function) of a graph G is a 2-rD function f for which the set of all $v \in V(G)$ with $f(v) = \emptyset$ is independent. The outer independent 2-rainbow domination number $\gamma_{oir2}(G)$ is the minimum weight of an OI2-rD function of G. In this paper, we study the OI2-rD number of graphs. We give the complexity of the problem OI2-rD of graphs and present lower and upper bounds on $\gamma_{oir2}(G)$. Moreover, we characterize graphs with some small or large OI2-rD numbers and we also bound this parameter from above for trees in terms of the order, leaves and the number of support vertices and characterize all trees attaining the bound. Finally, we show that any ordered pair (a, b) is realizable as the vertex cover number and OI2-rD numbers of some non-trivial tree if and only if $a+1 \leq b \leq 2a$.

Keywords: outer-independent rainbow domination, $K_{1,r}$-free graphs, trees.

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1. INTRODUCTION AND PRELIMINARIES

For notation and terminology which are not given here, we refer to [11]. Let $G = (V(G), E(G))$ be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The *open neighborhood* of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V(G)$, its open neighborhood is $N(S) = \cup_{v \in S} N(v)$ and its closed neighborhood is the set $N[S] = N(S) \cup S$. The *maximum* (*minimum*) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$), respectively. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the minimum length of a (u, v)-path in G. The diameter of G, $diam(G)$, is the maximum distance among all the pairs of vertices in G. In a graph G, the length of a longest cycle is called its *girth*. If G has no cycle, the girth of G is defined to be infinite. A support vertex is called *strong support vertex* if it is adjacent to at least two leaves.

A set $S \subseteq V(G)$ is called a *dominating set* in G if $N[S] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set $S \subseteq V(G)$ is *independent* if no two vertices in S are adjacent. The maximum cardinality of an independent set in G ($\alpha(G)$) is said to be the *independence number* of G. A *vertex cover* Q of a graph G is a set of vertices that every edge has an end point in Q. The minimum cardinality of a vertex cover is denoted by $\beta(G)$. The *corona* of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

A function $f : V(G) \to \mathcal{P}(\{1,2\})$ is a 2*-rainbow dominating function* (2-rD function) if for every vertex x with $f(x) = \emptyset$, $f(N(x)) = \{1,2\}$. The 2*-rainbow domination number* $\gamma_{r2}(G)$ is the minimum weight of $\sum_{v \in V(G)} |f(v)|$ taken over all 2-rD functions. A 2-rD function f is an *independent* 2*-rainbow dominating function* (I2-rD function) if no two vertices assigned nonempty sets are adjacent. The weight of an I2-rD function f is the value $w(f) = \sum_{v \in V(G)} |f(v)|$. The *independent* 2*-rainbow domination number* $\gamma_{i_{r2}}(G)$ is the minimum weight of an I2-rD function of G. The concept of rainbow domination was introduced by Bresar *et al.* [1] and inspired the works [2,3] and [12]. It is worth mentioning that since 2004, many papers have been published in which some new variations such as rainbow domination, connected rainbow domination, total rainbow domination, independent rainbow domination and rainbow domination of directed graphs were introduced [6,8], and the relation between rainbow domination and domination, Roman domination and double Roman domination, independent double Roman domination have been investigated [9,10].

Definition 1.1 ([7])**.** A function $f : V(G) \to \mathcal{P}(\{1,2\})$ is an *outer independent* 2*-rainbow dominating function* (OI2-rD function) of G if f is a 2-rD function and the set of vertices with weight $\emptyset$ is independent. The *outer independent* 2*-rainbow domination number* (OI2-rD number) $\gamma_{oir2}(G)$ is the minimum weight of an OIk-rD function of G. An OI2-rD function of weight $\gamma_{oir2}(G)$ is called a $\gamma_{oir2}(G)$-function.

For an OI2-rD function f, we let $V_\emptyset$, $V_{\{1\}}$, $V_{\{2\}}$ and $V_{\{1,2\}}$ stand for the set of vertices assigned with $\emptyset$, $\{1\}$, $\{2\}$ and $\{1,2\}$ under f. Since these four sets determine f, we can equivalently write $f = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$. Note that

$$w(f) = |V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}|.$$

This paper is organized as follows: We study some preliminary results on OI2-rD number of graphs in Section 2. In Section 3, we characterize graphs G with some small or large OI2-rD numbers. We give the complexity of the problem OI2-rD of graphs in Section 4. In Section 5 we study the lower and upper bounds on $\gamma_{oir2}(G)$. In Section 6 we also bound this parameter from above for trees in terms of the order, leaves and the number of support vertices and characterize all trees attaining the bound. Finally, we show that any ordered pair (a, b) is realizable as the vertex cover number and OI2-rD numbers of some non-trivial tree if and only if $a+1 \leq b \leq 2a$. We end the paper with conclusions and problems.

2. PRELIMINARY RESULTS

In this section, we obtain some basic results and give the exact formulas for the OI2-rD numbers for some well-known graphs. We first observe that $\gamma_{oir2}(G)$ is well defined for all graphs, because every graph G has a trivial OI2-rD function $v \to \{1\}$ for every $v \in V(G)$.

We first present the exact formulas for the OI2-rD numbers for paths, cycles, complete, and complete multipartite graphs. We only obtain the OI2-rD numbers of cycles and left others.

Observation 2.1. *For $n \geq 1$, $\gamma_{oir2}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Observation 2.2. *For $n \geq 3$, $\gamma_{oir2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.*

Proof. Let $f = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$ be a $\gamma_{oir2}(C_n)$-function. It is clear that $|V_0| \leq \lfloor \frac{n}{2} \rfloor$. So,
$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \leq \gamma_{oir2}(C_n).$$
Let $n \in \{4k, 4k+1, 4k+3\}$. Then the function f with $f(v_{4t+1}) = \{1\}$, $f(v_{4t+3}) = \{2\}$ and $f(v_{4t}) = f(v_{4t+2}) = \emptyset$ is a $\gamma_{oir2}(C_n)$-function.

Now let $n = 4k+2$. Then the function f with $f(v_{4t+1}) = \{1\}, f(v_{4t+3}) = \{2\}$ for $4t+1 \neq n-1$, $f(v_{4t}) = f(v_{4t+2}) = \emptyset$ and $f(v_{n-1}) = \{1,2\}$, is a $\gamma_{oir2}(C_{4k+2})$-function. All in all, we observe that $w(f) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$. □

Observation 2.3. *The following statements hold.*

(a) *For $n \geq 3$, $\gamma_{oir2}(K_n) = n - 1$.*

(b) *For $m \leq n$, $\gamma_{oir2}(K_{m,n}) = \begin{cases} 2 & \text{if } m = 1, \\ m & \text{if } m \geq 2. \end{cases}$*

(c) *Let $k \geq 3$ and $K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph with $1 \leq n_1 \leq n_2 \leq \ldots \leq n_k$. Then, $\gamma_{oir2}(K_{n_1,n_2,\ldots,n_k}) = \sum_{i=1}^{k-1} n_i$.*

3. GRAPHS G WITH SMALL OR LARGE $\gamma_{oir2}(G)$

Our aim in this section is to characterize the connected graphs with small or large OI2-rD numbers. It is obvious that, for any connected graph G of order $n \geq 2$, $2 \leq \gamma_{oir2}(G) \leq n$.

We first give the characterizations of all connected graphs G for which $\gamma_{oir2}(G) \in \{2, 3\}$.

Proposition 3.1. *Let G be a connected graph. Then $\gamma_{oir2}(G) = 2$ if and only if $G \in \{K_{1,n},\ K_{2,n}\}$ or $G = K^*_{2,n}$ which is obtained from $K_{2,n}$ by joining two vertices in its 2-vertex partite set, where $n \geq 2$.*

Proof. Clearly,

$$\gamma_{oir2}(K_{1,n}) = \gamma_{oir2}(K_{2,n}) = \gamma_{oir2}(K^*_{2,n}) = 2.$$

Conversely, let $\gamma_{oir2}(G) = 2$ and f be a $\gamma_{oir2}(G)$-function. Then, one of the following situations holds.

(a) There exists a vertex v for which $f(v) = \{1, 2\}$. In such a case the other vertices are independent and adjacent to v, and f assigns $\emptyset$ to them. Therefore $G = K_{1,n}$.
(b) There exist two vertices v and u for which $(f(u), f(v)) = (\{1\}, \{2\})$ and $f(w) = \emptyset$ for the other vertices. Note that the vertices in $V(G) \setminus \{u, v\}$ are independent and adjacent to both u and v. Therefore, $G \in \{K_{2,n}, K^*_{2,n}\}$. □

We now give the characterization of all connected graphs G with $\gamma_{oir2}(G) = 3$. To this aim, we define the families $\mathcal{R}_i$, $1 \leq i \leq 7$, of graphs G as follows. We first fix some notation. For given vertices x, y and z, we set

$$\begin{aligned} V_x &= \{v \in V(G) \mid N(v) = \{x\}\}, \\ V_{x,y} &= \{v \in V(G) \mid N(v) = \{x, y\}\}, \\ V_{x,y,z} &= \{v \in V(G) \mid N(v) = \{x, y, z\}\}, \end{aligned}$$

where $V_x \cup V_{x,y} \cup V_{x,y,z}$ is an independent set.

$\mathcal{R}_1$: We begin with two nonadjacent vertices x and y. Then, we add the nonempty sets V_x and $V_{x,y}$.

$\mathcal{R}_2$: The family of all graphs G obtained by adding the edges xy to the graphs in the family $\mathcal{R}_1$.

$\mathcal{R}_3$: We begin with three vertices x, y and z. Then one of the following conditions holds. (a_3) $V_{x,y}, V_{y,z} \neq \emptyset$, (b_3) only one of $V_{x,y}$ and $V_{y,z}$ is empty set and $V_{x,y,z} \neq \emptyset$, (c_3) $V_{x,y} \cup V_{y,z} = \emptyset$ and $|V_{x,y,z}| \geq 3$.

$\mathcal{R}_4$: We begin with an edge xy and a vertex z. Then (a_4) $V_{y,z} \neq \emptyset$, or (b_4) $V_{y,z} = \emptyset$ and $V_{x,y,z} \neq \emptyset$.

$\mathcal{R}_5$: We begin with a path xyz. Then (a_5) $V_{x,y} \cup V_{y,z} \neq \emptyset$, or (b_5) $V_{x,y} \cup V_{y,z} = \emptyset$ and $|V_{x,y,z}| \geq 2$.

$\mathcal{R}_6$: We begin with the edges xy and xz. Then one of the following conditions holds. (a_6) $V_{x,y} \neq \emptyset$, (b_6) $V_{x,y} = \emptyset$ and both $V_{y,z}$ and $V_{x,y,z}$ are nonempty set, (c_6) $V_{x,y} = \emptyset$ and $|V_{x,y,z}| \geq 2$.

$\mathcal{R}_7$: We begin with a cycle $xyzx$. Then (a_7) $V_{x,y}, V_{y,z} \neq \emptyset$, or (b_7) $V_{x,y}$ or $V_{y,z}$ equals empty set and $V_{x,y,z} \neq \emptyset$.

Theorem 3.2. *Let G be a connected graph of order n. Then, $\gamma_{oir2}(G) = 3$ if and only if $G \in \bigcup_{i=1}^{7} \mathcal{R}_i$.*

Proof. If $G \in \mathcal{R}_1 \cup \mathcal{R}_2$, then the function g with $g(x) = \{1,2\}$, $g(y) = \{1\}$ and $g(v) = \emptyset$ for the other vertices is an OI2-rD function of G with weight $\gamma_{oir2}(G) = 3$. If $G \in \bigcup_{i=3}^{7} \mathcal{R}_i$, then the function g with $g(x) = g(z) = \{1\}$, $g(y) = \{2\}$ and $g(v) = \emptyset$ for the other vertices defines an OI2-rD function of G with weight $\gamma_{oir2}(G) = 3$.

Conversely, let $f : V(G) \to \mathcal{P}(\{1,2\})$ be a minimum OI2-rD function with weight $\omega(f) = 3$. We consider two cases depending on $V_{\{1,2\}}(= \{v \in V(G) \mid f(v) = \{1,2\}\})$.

Case 1. Let $V_{\{1,2\}} \neq \emptyset$. Then, there exists a unique vertex x with weight $\{1,2\}$ under f and one vertex y, with $|f(y)| = 1$. Note that the other vertices are assigned $\emptyset$ under f and belong to $V_x \cup V_{x,y}$. If $V_x = \emptyset$, then $\gamma_{oir2}(G) = 2$ which is impossible. Also, if $V_{x,y} = \emptyset$, then either $\gamma_{oir2}(G) = 2$ or G is disconnected, a contradiction. Therefore, $G \in \mathcal{R}_1 \cup \mathcal{R}_2$.

Case 2. Let $V_{\{1,2\}} = \emptyset$. We assume that $(f(x), f(y), f(z)) = (\{1\}, \{2\}, \{1\})$ for some vertices x, y and z, and $f(v) = \emptyset$ for the other vertices v. Note that the other vertices v are independent and adjacent to y, necessarily.

Let first $G[\{x,y,z\}]$ be edgeless. If both $V_{x,y}$ and $V_{y,z}$ are nonempty, then $G \in \mathcal{R}_3$. If both $V_{x,y}$ and $V_{y,z}$ are empty, then $|V_{x,y,z}| \geq 3$, otherwise $\gamma_{oir2}(G) = 2$ or G is disconnected. If only one of $V_{x,y}$ and $V_{y,z}$, say $V_{x,y}$, is empty, then $V_{x,y,z} \neq \emptyset$, for otherwise G would be disconnected. This shows that $G \in \mathcal{R}_3$.

Let y be adjacent to only one of x and z, say x, and $xz \notin E(G)$. If $V_{y,z} = \emptyset$, then $V_{x,y,z} \neq \emptyset$, for otherwise G is disconnected. So, $G \in \mathcal{R}_4$.

Let $G[\{x,y,z\}]$ be isomorphic to the path xyz. Let $V_{x,y} \cup V_{y,z} = \emptyset$. If $|V_{x,y,z}| \leq 1$, then $\gamma_{oir2}(G) = 2$ which is impossible. Therefore, $|V_{x,y,z}| \geq 2$.

Suppose that $xy, xz \in E(G)$ and $yz \notin E(G)$. Suppose that $V_{x,y} = \emptyset$. Then we must have $V_{x,y,z} \neq \emptyset$, for otherwise $\gamma_{oir2}(G) = 2$. If $|V_{x,y,z}| = 1$, then $V_{y,z} \neq \emptyset$, for otherwise we have again $\gamma_{oir2}(G) = 2$. Therefore, $V_{y,z} \neq \emptyset$. This implies that $G \in \mathcal{R}_6$.

Finally, suppose that $G[\{x,y,z\}]$ is isomorphic to the cycle $xyzx$. Suppose that at least one of $V_{x,y}$ and $V_{y,z}$ is empty set. Then $V_{x,y,z}$ is nonempty set, for otherwise $\gamma_{oir2}(G) = 2$. This completes the proof. □

In what follows, we characterize all graphs G of order n with large $\gamma_{oir2}(G) \in \{n-1, n\}$.

Proposition 3.3. *Let G be a graph of order n. Then $\gamma_{oir2}(G) = n$ if and only if $G = mK_2 \cup \overline{K_t}$, where $n = 2m + t$.*

Proof. Let $G = mK_2 \cup \overline{K_t}$. Then it is clear $\gamma_{oir2}(G) = 2m + t = n$.

Conversely, let $\gamma_{oir2}(G) = n$. Let G have a vertex v of degree at least 2. Then by assigning $\emptyset$ to v, $\{1\}$ to one neighbor of v and $\{2\}$ to other vertices, we obtain $\gamma_{oir2}(G) \leq n - 1$, a contradiction. Therefore, every vertex of G has degree at most 1, that is, $G = mK_2 \cup \overline{K_t}$. □

Let K_n^t be a graph obtained from complete graph K_n by joining t leaves to t vertices of K_n, where $0 \leq t \leq n$. We note that $K_n^0 = K_n$ and $K_n^n = K_n \circ K_1$. We also observe that if G is a graph obtained from H by joining t leaves to the t vertices of H, then $\gamma_{oir2}(G) \leq \gamma_{oir2}(H) + t$.

Theorem 3.4. *Let G be a connected graph of order n. Then $\gamma_{oir2}(G) = n - 1$ if and only if $G \in \{P_3, P_4, K_m^t\}$ for some $m \geq 3$ and $0 \leq t \leq m$.*

Proof. The equality is trivial for $G \in \{P_3, P_4\}$. Let G be isomorphic to K_m^t for some $m \geq 3$ and $0 \leq t \leq m$. It is not difficult to see that the function f assigning $\emptyset$ to only one vertex u of K_m, $\{1\}$ to the leaf adjacent to it (if any) and one vertex $v \in V(K_m^t)$ different from u, and $\{2\}$ to the other vertices is an OI2-rD function with weight $\gamma_{oir2}(K_m^t) = m + t - 1 = n - 1$.

Conversely, let $G \neq P_3, P_4$ and $\gamma_{oir2}(G) = n - 1$. Let $f = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$ be a $\gamma_{oir2}(G)$-function with weight $n - 1$. Suppose that u and v are two distinct vertices of G with $\deg(u), deg(v) \geq 2$. Suppose that $uv \notin E(G)$. Then, we deal with one of the following possibilities depending of $N(u)$ and $N(v)$.

(a) Let $N(u) \cap N(v) \neq \emptyset$ and let $x \in N(u) \cap N(v)$. Then the assignment $(g(u), g(v), g(x)) = (\emptyset, \emptyset, \{1\})$ and $g(w) = \{2\}$ for the other vertices w defines an OI2-rD function of G with weight $n - 2$.
(b) Let $N(u) \cap N(v) = \emptyset$. Let $x, y \in N(u)$ and $x', y' \in N(v)$. It is easily seen that the assignment

$$(h(u), h(v), h(x), h(y), h(x'), h(y')) = (\emptyset, \emptyset, \{1\}, \{2\}, \{1\}, \{2\})$$

and $h(w) = \{2\}$ for the other vertices w defines an OI2-rD function of G with weight $n - 2$.

We deduce from the the both above situations that $\gamma_{oir2}(G) \leq n - 2$, which is a contradiction. Therefore, $uv \in E(G)$. This shows that the subgraph induced by the vertices of degrees at least two is a complete graph K_m. Therefore, G is obtained from K_m by joining some leaves to some vertices of K_m. Now if $m = 1$, then $G \cong K_{1,t}$ for some $t \geq 2$. In such a case $G \cong P_3$ or $\gamma_{oir2}(G) < n - 1$ which are impossible. If $m = 2$, then G is isomorphic to a double star $S_{p,q}$. In such a case $G \cong P_4$ or $\gamma_{oir2}(G) < n - 1$ which are again impossible. Therefore, $m \geq 3$. Suppose now that there exists a vertex u of K_m which is adjacent to two leaves x and y. Let $v \in V(K_m) \setminus \{u\}$. Then

$$(k(v), k(x), k(y), k(u)) = (\emptyset, \emptyset, \emptyset, \{1, 2\})$$

and $k(w) = \{1\}$ for the other vertices w defines an OI2-rD function of G with weight $n - 2$. This is a contradiction. Therefore, every vertex of K_m is adjacent to at most one leaf. This completes the proof. □

Let G be a graph of order n. If $\gamma_{oir2}(G) = n - 2$ and $f = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$ is a $\gamma_{oir2}(G)$-function, then

$$\begin{aligned}
&|V_\emptyset| + |V_{\{1\}}| + |V_{\{2\}}| + |V_{\{1,2\}}| = n, \\
&|V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}| = n - 2, \\
&|V_\emptyset| = |V_{\{1,2\}}| + 2.
\end{aligned}$$

Let M be a matching of K_n with $|M| \geq 1$, and let $K_n - M$ be a graph obtained from the complete graph K_n by removing the edges in M. Let $(K_n - M)^t$ be a graph obtained from the graph $K_n - M$ by joining t leaves to t vertices of $K_n - M$, where $0 \leq t \leq \lceil \frac{n}{2} \rceil$. Let P_n^+, (C_n^+) be a graph obtained from P_n (C_n) by attaching a leaf to any vertex of P_n (C_n) and let for $(m \geq 4)$, P_m^*, (C_m^*) be a graph obtained from P_m (C_m) by attaching some leaves (at least 2 vertices) to some vertices (one leaf to a vertex) of P_m (C_m).

Proposition 3.5. *Let G be a connected graph of order $n \geq 5$. If*

$$G \in \{P_5, P_5^+, P_6, C_5, C_5^+, C_6, P_4^*, C_4^*, (K_m - M)^t\},$$

where $0 \leq t \leq \lceil \frac{m}{2} \rceil$ and $t + m = n$, then $\gamma_{oir2}(G) = n - 2$.

Proof. It is easily seen that if $G \in \{P_5, P_5^+, P_6, C_5, C_5^+, C_6, P_4^*, C_4^*\}$, then $\gamma_{oir2}(G) = n - 2$. Let $G = (K_m - M)^t$. Let $t = 0$ and M be a matching of K_n. Then, every vertex of $\delta(K_n - M) = n - 2$. Therefore $\gamma_{oir2}(G) = n - 2$. Let $t \geq 1$. Since $|M| \geq 1$ we could assign the value $\emptyset$ only to two vertices of $K_n - M$. If $v_i u_i$ is a pendant edge where v_i is a vertex of $K_n - M$ and u_i is a leaf, then we must assign $\{1, 2\}$ to v_i and $\emptyset$ to u_i or we must assign $\{1\}$ or $\{2\}$ to v_i and $\{1\}$ to u_i. Therefore, for any t attached vertices, the weight of $G = (K_n - M)^t$ will be increased t units. That is $w((K_n - M)^t) = n - 2 + t$. Therefore, for the given graphs G constructed as above of order $|V(G)|$, $\gamma_{oir2}(G) = |V(G)| - 2$. $\square$

The converse of the Proposition 3.5 may be easily solved. We pose it as a problem at the end of the paper.

4. COMPLEXITY

We consider in this section the decision problem associated with the OI2-rD functions. We first consider the problem of deciding whether a graph G has the OI2-rD number at most a given integer. That is stated in the following decision problem.

OI2-RD problem
INSTANCE: A graph G and an integer $k \leq |V(G)|$.
QUESTION: Is $\gamma_{oir2}(G) \leq k$?

Our aim is to show that the problem is NP-complete for the planar graphs with maximum degree at most four. To this end, we make use of the well-known INDEPENDENCE NUMBER PROBLEM (IN problem) which is known to be NP-complete from [5].

IN problem
INSTANCE: A graph G and an integer $k \leq |V(G)|$.
QUESTION: Is $\alpha(G) \geq k$?

Moreover, the problem above remains NP-complete even when restricted to triangle-free graphs and some planar graphs. Indeed, we have the following result.

Theorem 4.1 ([5]). *The IN problem is NP-complete even when restricted to triangle-free graphs and planar graphs of maximum degree at most three.*

Theorem 4.2. *The OI2-rD problem is NP-complete even when restricted to triangle-free graphs and planar graphs with maximum degree at most four.*

Proof. Let G be with the set of vertices $V(G) = \{v_1, \ldots, v_n\}$ be a triangle-free graph or a planar graph with maximum degree $\Delta(G) \leq 3$. For any $1 \leq i \leq n$, we add a copy of the path P_3 with central vertex u_i. We now construct a graph G' by joining v_i to u_i, for each $1 \leq i \leq n$. Clearly,

(a) if G is triangle-free, then G' is triangle-free, as well:
(b) if G is a planer graph with $\Delta(G) \leq 3$, then G' is a planer graph with $\Delta(G') \leq 4$.

Moreover, $|V(G')| = 4n$.

Let f be $\gamma_{oir2}(G')$-function. Since u_i is adjacent to two leaves, the sum of cardinalities of the assigned sets to u_i and its two leaves under f must be two. So, without loss of generality, we may consider that $f(u_i) = \{1, 2\}$, and that f assigns $\emptyset$ to both leaves adjacent to u_i for each $1 \leq i \leq n$. Since $V_\emptyset$ is independent, the number of vertices $v_i \in V(G)$ which can be assigned $\emptyset$ under f is at most $\alpha(G)$. Furthermore, the other vertices of $V(G)$ are assigned non-empty sets under f. Consequently, we obtain that

$$\gamma_{oir2}(G') \geq 2n + (n - \alpha(G)) = 3n - \alpha(G).$$

On the other hand, let I be an $\alpha(G)$-set. It is easy to observe that the function

$$g(v) = \begin{cases} \{1,2\} & \text{if } \ v \in \{u_1, \ldots, u_n\}, \\ \emptyset & \text{if } \ v \text{ is a leaf or } v \in I, \\ \{1\} \text{ or } \{2\} & \text{otherwise.} \end{cases}$$

is an OI2-rD function of G' with weight $3n - \alpha(G)$, which leads to the equality $\gamma_{oir2}(G') = 3n - \alpha(G)$. Now, by taking $j = 3n - k$, it follows that $\gamma_{oir2}(G') \leq j$ if and only if $\alpha(G) \geq k$, which completes the reduction. Since the IN problem is NP-complete for both triangle-free, and the planar graphs of maximum degree at most three, we deduce that the OI2-rD problem is NP-complete for triangle-free graphs as well, and it is NP-complete for planar graphs of maximum degree at most four. $\square$

As a consequence of Theorem 4.2, we conclude that the problem of computing the OI2-rD number is NP-hard even when restricted to planar graphs with maximum degree at most four and triangle-free graphs.

5. BOUNDS

Obviously, every OI2-rD function of a graph is a 2-rainbow dominating function, and so $\gamma_{oir2}(G) \geq \gamma_{r2}(G)$ holds for every graph G. The equality occurs for stars, however the difference between these two parameters can be arbitrarily large. For example, for the complete graphs of large order, we have $\gamma_{oir2}(K_n) = n - 1$, while $\gamma_{r2}(K_n) = 2$.

It is clear that if H is an induced subgraph of a graph G of order n, then

$$\gamma_{oir2}(G) \leq n - |V(H)| + \gamma_{oir2}(H).$$

Now by considering H as a diametral path on $diam(G) + 1$ vertices in G (in the case when G is connected) and making use of Observation 2.1, we have

$$\gamma_{oir2}(G) \leq (n - diam(G) - 1) + \gamma_{oir2}(H) = n - \left\lfloor \frac{diam(G)}{2} \right\rfloor.$$

Note that the family of all paths achieve this bound.

In what follows, we shall see the upper bound for outer independent 2-rainbow domination number in terms of independence number or vertex cover number. For this aim we introduce a family of graphs as follows. Let $\mathcal{G}$ be family of graphs G constructed from a graph H,

(a) by attaching at least two leaves to the every vertex of H, and possibly for some vertices x, $deg_{G-H}(x) \geq 3$, or,
(b) by attaching at least one leaf to the every vertex of H, moreover, for any vertex x (possibly other than isolated vertex in H), $deg_{G-H}(x) \geq 2$, or for two adjacent vertices in H like u, v we must have $deg_{G-H}(u) + deg_{G-H}(v) \geq 4$, or any two non adjacent vertices in H like u, v must be neighbored by a vertex w in G, and $deg_{G-H}(u) + deg_{G-H}(v) \geq 5$, or beside, any k independent vertices in H have at least k independent neighbors in $G - H$ of degree k in G.

Moreover, in both of parts (a) and (b), $G - H$ is an independent set.

Proposition 5.1. *For any graph G of order n with no isolated vertices,*

$$\gamma_{oir2}(G) \leq 2\beta(G).$$

Furthermore, the equality holds if and only if $G \in \mathcal{G}$.

Proof. Let S be a maximum independent set in G of size $\alpha(G)$. Let $f : V(G) \to \mathcal{P}(\{1,2\})$ be a function with $f(v) = \{1,2\}$ for $v \in V - S$, and $f(x) = \emptyset$ for otherwise. It is easy to see that f is an OI2-rD function of G with weight $2(n - \alpha(G))$. Since $\alpha(G) + \beta(G) = n$ (the well known Gallai theorem [4]), the upper bound follows.

For equality, let $G \in \mathcal{G}$ and G satisfy in the conditions (a) and (b). Then we must assign $\{1,2\}$ to every vertex of H and $\emptyset$ to the all vertices of $V(G) \setminus V(H)$. On the other hand, it is readily seen that $\alpha(G) = |V(G) \setminus V(H)|$. Therefore

$$\gamma_{oir2}(G) = 2(n(G) - \alpha(G)) = 2\beta(G).$$

Conversely, let the equality hold. Let S be an $\alpha(G)$-set and H be the subgraph induced by $V(G) \setminus S$. If a vertex $v \in H$ has now leaf in G, then by assigning $\{1\}$ to v, $\{1,2\}$ to the other vertices in H and $\emptyset$ to vertices of $G - H$, achieve a contradiction. Thus it is easy to see that, at least one leaf is adjacent to each vertex of $G - S$. If each vertex in H has at least two leaves in G, then G satisfies in condition (a). Let v be a vertex with only one leaf in $G-H$. If v does not satisfy in condition (b), there are some cases.

(1) If $deg_{G-H}(v)=1$, then by assigning $\{1,2\}$ to $V(H-v)$ and $\{1\}$ to the leaf adjacent to v in G and $\emptyset$ to others, we have an OI2-rD function with weight at most $2\beta(G)-1$, a contradiction.

(2) If there exists two adjacent vertices in H like u,v such that $deg_{G-H}(u)+deg_{G-H}(v)\leq 3$, and let $deg_{G-H}(u)=1$ and $deg_{G-H}(v)\leq 2$, then by assigning $\{1,2\}$ to $V(H-u)$, $\{1\}$ to the leaf adjacent to u and $\emptyset$ to others, we have an OI2-rD function with weight at most $2\beta(G)-1$, a contradiction.

(3) If there exists two non adjacent vertices in H like u,v that are neighbored by a vertex w in G, and $deg_{G-H}(u)+deg_{G-H}(v)\leq 4$, then by assigning $\{1,2\}$ to $V(H-\{u,v\})$, $\{1\}$ to the leaves adjacent to u and v and $\{2\}$ to w and $\emptyset$ to others, we have an OI2-rD function with weight at most $2\beta(G)-1$, a contradiction.

(4) If there exist k independent vertices $\{v_1,v_2,\ldots,v_k\}$ in H with at most $k-1$ independent neighbors $\{u_1,u_2,\ldots,u_{k-1}\}$ in $G-H$ of degree k in G, by assigning $\{1\}$ to some of (not all) u_is, $\{2\}$ to other of $k-1$ u_is, $\{1\}$ to each leaf adjacent to v_j in $G-H$, $\{1,2\}$ to the vertices of $V(H-\{v_1,v_2,\ldots,v_k\})$ and $\emptyset$ to the other vertices, we have an OI2-rD function with weight at most $2\beta(G)-1$, a contradiction. Thus, if the equality holds, then $G\in\mathcal{G}$. Therefore, the proof ends. □

Let G be a connected graph with finite girth g. As an immediate consequence of Observation 2.2, we have

$$\gamma_{oir2}(G)\leq n-\left\lfloor\frac{g}{2}\right\rfloor+\left\lceil\frac{g}{4}\right\rceil-\left\lfloor\frac{g}{4}\right\rfloor.$$

Furthermore, this bound is sharp for cycles.

Theorem 5.2. *Let G be a $K_{1,r}$-free graph of order n with s' strong support vertices. Then*

$$\gamma_{oir2}(G)\geq\frac{2(n+s')}{1+r}$$

and this bound is sharp. In particular, we have the sharp lower bound

$$\gamma_{oir2}(G)\geq\frac{2(n+s')}{\Delta+2}.$$

Proof. Let f be a $\gamma_{oir2}(G)$-function. We define A as $V_\emptyset\cap N(V_{\{1,2\}})$. Since G is $K_{1,r}$-free and $V_\emptyset$ is independent, every vertex in $V_{\{1,2\}}$ is adjacent to at most $r-1$ vertices in A. Thus,

$$|A|\leq(r-1)|V_{\{1,2\}}|.$$

On the other hand, every vertex in $V_\emptyset\setminus A$ has at least two neighbors in $V_{\{1\}}\cup V_{\{2\}}$. Therefore,

$$2|V_\emptyset\setminus A|\leq(r-1)(|V_{\{1\}}|+|V_{\{2\}}|).$$

Moreover, we may assume that all the strong support vertices belong to $V_{\{1,2\}}$. This shows that $|V_{\{1,2\}}| \geq s'$. We now have

$$\begin{aligned} 2(n - \gamma_{oir2}(G) + s') &\leq 2(n - |V_{\{1\}}| - |V_{\{2\}}| - |V_{\{1,2\}}|) = 2|V_\emptyset| \\ &= 2|A| + 2|V_\emptyset \setminus A| \\ &\leq (r-1)(|V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}|) \\ &= (r-1)\gamma_{oir2}(G). \end{aligned}$$

So,

$$\gamma_{oir2}(G) \geq \frac{2(n+s')}{1+r}.$$

For sharpness consider $\overline{K_p} \circ \overline{K_{r-1}}$ for $r \geq 2$. Then $\gamma_{oir2}(G) = 2p$, $n = pr$ and $s' = p$. This implies that

$$\gamma_{oir2}(G) = \frac{2(n+s')}{r+1}.$$

For the other family of sharpness graph, let G be a graph of order $r+1 \geq 4$ with two vertices v, u of degree $r-1$ and $r-1$ vertices of degree 2 for which, each vertex of degree 2 is adjacent to u and v. Then G is $K_{1,r}$-free graph and

$$2 = \gamma_{oir2}(G) = \frac{2(n+s')}{r+1}.$$

Let $r = 3$ (in such a case G is claw-free). We begin with a cycle $v_1 v_2 \ldots v_{2p} v_1$. Add $2p$ new vertices $u_1, \ldots, u_{2p}$ and join u_i to both v_i and v_{i+1} (mod $2p$), for $1 \leq i \leq 2p$. It is easy to see that the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(v_{2i-1}) = \{1\}$, $f(v_{2i}) = \{2\}$ for $1 \leq i \leq p$, and $f(u_j) = \emptyset$ for $1 \leq j \leq 2p$ is an OI2-rD function of G with weight $\gamma_{oir2}(G) = \frac{n}{2}$.

Finally, for any graph G with maximum degree Δ, the graph G is a $(\Delta+1)$-free. Therefore, using first part, we have the desired result. □

6. TREES

Our aim in this section is to determine some bounds on the OI2-rD number of trees. We bound the OI2-rD number of trees from above and characterize all trees attaining the bound. Let $S(T)$ and $L(T)$ be the set of support vertices and the set of leaves of a tree T, respectively. Let T be a tree with $s(T) = |S(T)|$ and $l(T) = |L(T)|$. In order to characterize all trees T attaining the upper bound given in the next theorem, we introduce a partition of $V(T)$ as follows. Let T' be a tree as a component of the forest F obtained from T by removing all leaves and support vertices of T. Let v be a leaf of T'. Label each vertex of T' with its distance from v mod 2. This produces two sets $A(T') = \{u \mid d_{T'}(u,v) \text{ iseven}\}$ and $B(T') = \{u \mid d_{T'}(u,v) \text{ is odd}\}$ that partition the vertices of T'. We now have the partition $\mathcal{P} = \{S(T) \cup L(T), A(T'), B(T')\}_{T'}$ of the set of vertices of T. For the sake of convenience, we let $\mathcal{A}(T) = \cup_{T'} A(T')$ and $\mathcal{B}(T) = \cup_{T'} B(T')$. We can choose the vertex v for which, $|\mathcal{A}(T)| \geq |\mathcal{B}(T)|$.

Theorem 6.1. *Let T be a tree of order $n \geq 2$. Then,*

$$\gamma_{oir2}(T) \leq \left\lfloor \frac{n + 3s(T) - l(T)}{2} \right\rfloor.$$

This bound is sharp if and only if one of the following statements is fulfilled.

(1) *Every support vertex is strong and $|\mathcal{A}(T)| - |\mathcal{B}(T)| \leq 1$.*
(2) *here exists only one weak support vertex in T. Moreover, it is adjacent to a vertex in $\mathcal{A}(T)$ and $|\mathcal{A}(T)| - |\mathcal{B}(T)| = 1$.*

Proof. We make use of the notations which were introduced just before the theorem. Clearly, $n(F) = n(T) - s(T) - l(T)$. Suppose that f' assigns $\emptyset$ to the vertices in $\mathcal{A}(T)$, and $\{1\}$ or $\{2\}$ to the vertices in $\mathcal{B}(T)$ so that every vertex of T' which belongs to $\mathcal{A}(T)$ is adjacent to at least one vertex assigned with $\{1\}$ and at least one vertex assigned with $\{2\}$ under f'. Iterate this process for all components T' of F. We now define $g : V(T) \to \mathcal{P}(\{1, 2\})$ by

$$f(u) = \begin{cases} f'(u) & \text{if } u \in \mathcal{A}(T) \cup \mathcal{B}(T), \\ \{1, 2\} & \text{if } u \in S(T), \\ \emptyset & \text{if } u \in L(T). \end{cases}$$

It is not difficult to check that f is an OI2-rD function of T. Therefore,

$$\begin{aligned} \gamma_{oir2}(T) \leq \omega(f) &= \left\lfloor \frac{n(F)}{2} \right\rfloor + 2s(T) \\ &= \left\lfloor \frac{n - s(T) - l(T)}{2} \right\rfloor + 2s(T) \\ &= \left\lfloor \frac{n + 3s(T) - l(T)}{2} \right\rfloor. \end{aligned}$$

Let

$$\gamma_{oir2}(T) = \left\lfloor \frac{n + 3s(T) - l(T)}{2} \right\rfloor.$$

Suppose to the contrary that T has at least two weak support vertices x and y. If all vertices of T are leaves or support vertices, then

$$\gamma_{oir2}(T) < 2s(T) = \left\lfloor \frac{n + 3s(T) - l(T)}{2} \right\rfloor$$

which is impossible. Therefore, $S(T) \cup L(T) \subsetneqq V(T)$. Now let x' and y' be the leaves adjacent to x and y, respectively. Let F' be obtained from T by removing all leaves and support vertices except x and y. Similar to the process described for F, we have

$$\begin{aligned} \gamma_{oir2}(T) &\leq \left\lfloor \frac{n(F')}{2} \right\rfloor + 2(s(T) - 2) + 2 \\ &= \left\lfloor \frac{n - s(T) - l(T) + 2}{2} \right\rfloor + 2s(T) - 2 \\ &< \left\lfloor \frac{n + 3s(T) - l(T)}{2} \right\rfloor, \end{aligned}$$

a contradiction. Therefore, T has at most one weak support vertex.

Suppose to the contrary that the condition (1) does not hold. In what follows, we prove the condition (2). We first show that $|\mathcal{A}(T)| - |\mathcal{B}(T)| = 1$.

Let $|\mathcal{A}(T)| = |\mathcal{B}(T)|$. Since the condition (1) does not hold, it follows that T has precisely one weak support vertex x. Let F' be obtained from T by removing all leaves and support vertices except x and let x' be the unique leaf adjacent to x. We make use of a process similar to that presented for the proving the upper bound for $v = x$ as a leaf of a component T'' of F'. Therefore, $x \in A(T'')$. Since $|\mathcal{A}(T)| = |\mathcal{B}(T)|$, we may assume that

$$\sum_{T''} |A(T'')| = \sum_{T''} |B(T'')| + 1$$

taken over all components T'' of F'. Assigning $\emptyset$ to the vertices in $(\cup_{T''} A(T'') \cup L(T)) \setminus \{x'\}$, $\{1,2\}$ to all vertices in $S(T) \setminus \{x\}$, and $\{1\}$ or $\{2\}$ to the other vertices (so that each non-leaf vertex w assigned with $\emptyset$ is adjacent to at least one vertex assigned with $\{1\}$ and at least one vertex assigned with $\{2\}$) we obtain an OI2-rD function with weight

$$\Big\lfloor \frac{n - s(T) - l(T)}{2} \Big\rfloor + 2(s(T) - 1) + 1 < \Big\lfloor \frac{n + 3s(T) - l(T)}{2} \Big\rfloor,$$

a contradiction.

Now let $|\mathcal{A}(T)| - |\mathcal{B}(T)| \neq 1$. Since the equality $|\mathcal{A}(T)| = |\mathcal{B}(T)|$ is impossible, we may assume that $|\mathcal{A}(T)| \geq |\mathcal{B}(T)| + 2$. Assigning $\emptyset$ to the leaves and the vertices in $\mathcal{A}(T)$, $\{1,2\}$ to the support vertices, and $\{1\}$ or $\{2\}$ to the other vertices so that each non-leaf vertex w assigned with $\emptyset$ is adjacent to at least one vertex assigned with $\{1\}$ and at least one vertex assigned with $\{2\}$ defines an OI2-rD function of T with weight at most

$$|\mathcal{B}(T)| + 2s(T) < \Big\lfloor \frac{n + 3s(T) - l(T)}{2} \Big\rfloor$$

which contradicts the equality. Therefore, $|\mathcal{A}(T)| - |\mathcal{B}(T)| = 1$.

Now let x be the unique weak support vertex of T and x' be the leaf adjacent to x. Suppose to the contrary that x is adjacent to a vertex $z \in B(T')$, for some T'. Then, assigning $\{1,2\}$ to all strong support vertices, $\emptyset$ to all vertices in $\mathcal{A}(T) \cup (L(T) \setminus \{x'\}) \cup \{x\}$, and $\{1\}$ or $\{2\}$ to the other vertices so that each non-leaf vertex w assigned with $\emptyset$ is adjacent to at least one vertex assigned with $\{1\}$ and at least one vertex assigned with $\{2\}$ is an OI2-rD function of T with weight at most

$$\frac{n - s(T) - l(T) - 1}{2} + 2(s(T) - 1) + 1 < \Big\lfloor \frac{n + 3s(T) - l(T)}{2} \Big\rfloor,$$

which is again a contradiction. The above discussion shows that T satisfies the condition (2).

Conversely, let f be a $\gamma_{oir2}(T)$-function. Let the condition (1) hold. Since all support vertices are strong, we may assume that f assigns $\{1,2\}$ to all support vertices and $\emptyset$ to all leaves. Taking into account, the fact that $\mathcal{A}(T)$ and $\mathcal{B}(T)$ give us a partition of F into disjoint paths P_2, (possibly one of them is P_1), we observe that at most one vertex of each of the paths can be assigned $\emptyset$ by f. This shows that

$$\gamma_{oir2}(T) = \omega(f) \geq \frac{n - s(T) - l(T)}{2} + 2s(T) = \Big\lfloor \frac{n + 3s(T) - l(T)}{2} \Big\rfloor,$$

implying the equality.

Suppose now that the condition (2) holds. In such a case, the sets $\mathcal{A}(T)$ and $\mathcal{B}(T)$ give us a partition of F into disjoint paths P_2 and one singleton. Now consider the forest F. Suppose that $|\mathcal{A}(T)| = |\mathcal{B}(T)| + 1$. Then, at least $\frac{n-s(T)-l(T)-1}{2}$ vertices of F and at most $\frac{n-s(T)-l(T)+1}{2}$ vertices of F must be assigned $\emptyset$ under f. If exactly $\frac{n-s(T)-l(T)-1}{2}$ vertices of F are assigned $\emptyset$ by f (the vertices in $\mathcal{B}(T)$), then $f(x') \in \{\{1\},\{2\}\}$ and $f(x) = \emptyset$ since x is adjacent to a vertex in $\mathcal{A}(T)$ by (2). So,

$$\gamma_{oir2}(T) = \omega(f) = \frac{n - s(T) - l(T) + 1}{2} + 2(s(T) - 1) + 1 = \left\lfloor \frac{n + 3s(T) - l(T)}{2} \right\rfloor.$$

Now let $\frac{n-s(T)-l(T)+1}{2}$ vertices of F be assigned $\emptyset$ under f (the vertices in $\mathcal{A}(T)$). Since x is adjacent to a vertex in $\mathcal{A}(T)$, we have $(f(x), f(x')) = (\{1,2\}, \emptyset)$ or $f(x), f(x') \in \{\{1\},\{2\}\}$. Therefore,

$$\gamma_{oir2}(T) = \omega(f) = \frac{n - s(T) - l(T) - 1}{2} + 2s = \left\lfloor \frac{n + 3s(T) - l(T)}{2} \right\rfloor$$

implying the desired equality. This completes the proof. □

Lemma 6.2. *Let T be a tree of order n with s' strong support vertices. Then, $\gamma_{oir2}(T) \geq \beta(T) + s'$ and this bound is sharp.*

Proof. Let f be a $\gamma_{oir2}(T)$-function. We assume that f assigns $\{1,2\}$ to all strong support vertices of T and $\emptyset$ to all leaves adjacent to them. This shows that $|V_{\{1,2\}}| \geq s'$. On the other hand,

$$\gamma_{oir2}(T) = |V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}| \geq n - \alpha(T) + s' = \beta(T) + s'.$$

The bound is sharp for a tree T obtained from a tree T' by joining at least two leaves to each vertex of T'. □

Theorem 6.3. *For any tree T of order $n \geq 2$, $\beta(T) + 1 \leq \gamma_{oir2}(T) \leq 2\beta(T)$.*

Proof. The upper bound is deduced by Proposition 5.1. For the lower bound, we proceed by induction on the order n of T. The result is obvious when $n = 1$. On the other hand, $\gamma_{oir2}(K_{1,n-1}) = 2 = \beta(K_{1,n-1}) + 1$ for $n \geq 2$. Therefore, we may assume that $diam(T) \geq 3$. If T is isomorphic to a double star $S_{p,q}$, $1 \leq p \leq q$, we then deal with two possibilities. If $p = 1$, then $\gamma_{oir2}(S_{p,q}) = 3 = \beta(S_{p,q}) + 1$. If $p \geq 2$, then $\gamma_{oir2}(S_{p,q}) = 4 > 3 = \beta(S_{p,q}) + 1$. So, in what follows we assume that $diam(T) \geq 4$. This implies that $n \geq 5$.

Suppose that the lower bound holds for all trees T' of order $1 \leq n' < n$. Let T be a tree of order n. We consider two cases depending on the behavior of the support vertices of T.

Case 1. Suppose that there exists a strong support vertex of T. In such a case, the lower bound is an immediate result of Lemma 6.2.

Case 2. Suppose that all support vertices of T are weak. Let r and u be two vertices with $d(r,u) = diam(T)$. We root T at r. Let v be the parent of u and w be the parent of v. Note that in such a case we have $deg(v) = 2$. We now consider $T'' = T - u - v$. It is easy to check that $\beta(T) = \beta(T'') + 1$.

Let f be a $\gamma_{oir2}(T)$-function. Suppose that $f(w) = \{1, 2\}$. Then $f(v) = \emptyset$ and $f(u) = \{1\}$ or $\{2\}$, necessarily. This shows that $f'' = f\mid_{V(T'')}$ is an OI2-rD function of T'' with weight $\gamma_{oir2}(T) - 1$. Therefore,

$$\beta(T) = \beta(T'') + 1 \leq \gamma_{oir}(T'') \leq \gamma_{oir}(T) - 1$$

by the induction hypothesis.

Suppose that $f(w) = \emptyset$. Therefore, $|f(v)| + |f(u)| \geq 2$. We may assume, without loss of generality, that $f(v) = \{1, 2\}$ and $f(u) = \emptyset$. This shows that the assignment $(g(u), g(v), g(w)) = (\{1\}, \emptyset, \{2\})$ and $g(x) = f(x)$ for the other vertices defines a $\gamma_{oir}(T)$-function. Therefore, $g'' = g\mid_{V(T'')}$ is an OI2-rD function of T'' with weight $\gamma_{oir2}(T) - 1$. Therefore,

$$\beta(T) = \beta(T'') + 1 \leq \gamma_{oir2}(T'') \leq \gamma_{oir2}(T) - 1.$$

Let $f(w) = \{2\}$ ($f(w) = \{1\}$). This implies that $f(v) = \emptyset$ and $f(u) = \{1\}$ ($f(u) = \{2\}$). In such a case, we have again $\beta(T) + 1 \leq \gamma_{oir2}(T)$ similar to the case $f(w) = \emptyset$. This completes the proof of the lower bound. □

Our final result in this section is to show that every integer value in the range of Theorem 6.3 is realizable for trees, that is, all integer values between the lower and upper bounds in Theorem 6.3 are realizable.

Theorem 6.4. *An ordered pair (a, b) is realizable as the vertex cover number and OI2-rD numbers of some non-trivial tree if and only if $a + 1 \leq b \leq 2a$.*

Proof. We consider two cases.

Case 1. Let $b = a+1$. If $a = 1$, then it suffices to consider the star $K_{1,t}$ with $\beta(K_{1,t}) = 1$ and $\gamma_{oir2}(K_{1,t}) = 2$. So, we assume that $a \geq 2$. We consider the corona $T = K_{1,a-1} \circ K_1$. It is then easy to check that, $\beta(T) = a$ and $\gamma_{oir2}(T) = a + 1$.

Case 2. Let $b > a + 1$. Suppose that T' is obtained from the star $K_{1,a}$ by subdividing $a - 1$ edges. Let $v_1, \ldots, v_{a-1}$ be the new vertices. We now join one leaf to each vertex in $\{v_1, \ldots, v_{b-a-1}\}$. Let T be the obtained tree. It is easily verified that $\beta(T) = a$. On the other hand, the function f of G assigning $\{1, 2\}$ to $v_1, \ldots, v_{b-a-1}$, $\emptyset$ to the leaves adjacent to $v_1, \ldots, v_{b-a-1}$ and the weak support vertices different from the central vertex, $\{2\}$ to the central vertex, and $\{1\}$ to the other vertices is $\gamma_{oir2}(T)$-function with weight $2(b - a - 1) + a - (b - a - 1) + 1 = b$, as desired. This completes the proof. □

7. CONCLUSIONS AND PROBLEMS

(1) Prove or disprove: Let G be a connected graph of order at least 3. Then, $\gamma_{oir2}(G) = \gamma_{r2}(G)$ if and only if every induced subgraph of G is a union of cycles, paths, stars and complete bipartite graphs $K_{m,n},\ m, n \leq 4$.

(2) For a graph G, give some necessary and sufficient conditions in which $\gamma_{oir2}(G) = n-2$.
(3) Characterize all $K_{1,r}$-free (or at least claw-free) graphs for which the lower bound given in Theorem 5.2 holds with equality.

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