1.2 Fuzzy Relations

As crisp relations represent the association between elements of two or more sets, a fuzzy relation gives the extent of relationship between elements between two fuzzy sets. Zadeh [190] introduced fuzzy relations in 1965. Later, Zadeh [192], Kaufman, [91] and Rosenfeld [154] developed significant results. There are several applications for fuzzy relations. We have only a theoretical discussion about fuzzy relations in this section. We provide a formal definition below. Most of the contents of this section are based on Rosenfeld's work in 1975 [154].

If S represents a set, a fuzzy relation μ on S is a fuzzy subset of $S \times S$. In symbols, $\mu : S \times S \to [0, 1]$ such that $0 \leq \mu(x, y) \leq 1$ for all $(x, y) \in S \times S$. When μ takes the values 0 and 1 alone, it becomes the characteristic function of a relation on S. If R is a subset of S and P is a relation on S, then P becomes a relation on R only if $(x, y) \in P$ implies $x \in R$ and $y \in R$. If ζ and η are the characteristic functions of R and P respectively, then $\eta(x, y) = 1$ implies $\zeta(x) = \zeta(y) = 1$ for all $x, y \in R$. This is equivalent to the expression $\eta(x, y) \leq \zeta(x) \wedge \zeta(y)$ for all $x, y \in R$. Motivated by this, we have the definition of a fuzzy relation on a fuzzy subset as follows.

Definition 1.2.1 Let σ be a fuzzy subset of a set S and μ a fuzzy relation on S. Then μ is called a **fuzzy relation** on σ if $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in S$.

Definition 1.2.2 If S and T are two sets and σ and τ are fuzzy subsets of S and T, respectively, then a fuzzy relation μ from the fuzzy subset σ into the fuzzy subset τ is a fuzzy subset μ of $S \times T$ such that $\mu(x, y) \leq \sigma(x) \wedge \tau(y)$ for all $x \in S$ and $y \in T$.

It is interesting to see that for μ to be a fuzzy relation, the degree of membership of a pair of elements never exceeds the degree of membership of either of the elements. Later, while defining a fuzzy graph, this inequality allows us to organize the flow through an edge of a fuzzy graph in such a way that, it never exceeds the capacities of its end vertices. Also, μ^α is a relation from σ^α into τ^α for all $\alpha \in [0, 1]$ and as a consequence, μ^* becomes a relation from σ^* into τ^*.

In Definition 1.2.2, if $\sigma(x) = 1$ for all $x \in S$ and $\tau(y) = 1$ for all $y \in T$, then μ is called a fuzzy relation from S into T. Similarly, if $\sigma(x) = 1$ for all $x \in S$ in Definition 1.2.1, μ is said to be a fuzzy relation on S.

Definition 1.2.3 If σ is a fuzzy subset of a set S, the **strongest fuzzy relation** on σ is the fuzzy relation μ_σ defined by $\mu_\sigma(x, y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in S$.

Definition 1.2.4 For a fuzzy relation μ on S, the **weakest fuzzy subset** of S, on which μ is a fuzzy relation is σ_μ, defined by $\sigma_\mu(x) = \vee_{y \in S}(\mu(x, y) \vee \mu(y, x))$ for all $x \in S$.

Definition 1.2.5 Let $\mu : S \times T \to [0, 1]$ be a fuzzy relation from a fuzzy subset σ of S into a fuzzy subset τ of T and $\nu : T \times U \to [0, 1]$ is a fuzzy relation from the fuzzy subset ρ of T into a fuzzy subset η of U. Define $\mu \circ \nu : S \times U \to [0, 1]$ by $(\mu \circ \nu)(x, z) = \vee\{\mu(x, y) \wedge \nu(y, z) \mid y \in T\}$ for all $x \in S, z \in U$. Then $\mu \circ \nu$ is called the **max–min composition** of σ and τ.

The composition of any two fuzzy relations as in Definition 1.2.5 is always a fuzzy relation. But in the next result, we only consider two fuzzy relations defined on the same fuzzy set.

Proposition 1.2.6 *If μ and ν are fuzzy relations on a fuzzy set σ, then $\mu \circ \nu$ is a fuzzy relation on σ.*

Proof Let S be a set and σ be a fuzzy subset of S. Because μ and ν are fuzzy relations on σ, $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ and $\nu(y, z) \leq \sigma(y) \wedge \sigma(z)$ for all $x, y, z \in S$. Thus, $\mu(x, y) \wedge \nu(y, z) \leq \sigma(x) \wedge \sigma(y) \wedge \sigma(z) \leq \sigma(x) \wedge \sigma(z)$ for all $y \in S$ and hence, $(\mu \circ \nu)(x, z) = \vee_{y \in S}(\mu(x, y) \wedge \nu(y, z)) \leq \sigma(x) \wedge \sigma(z)$ for all $x, z \in S$. ■

Max–min composition is similar to matrix multiplication, where addition is replaced by $\vee$ and multiplication by $\wedge$. We can easily show that the composition of fuzzy relations is associative. So if we denote $\mu \circ \mu$ by μ^2, higher powers of the fuzzy relation μ^2, μ^3, and so on, can be easily defined. Define $\mu^\infty(x, y) = \vee\{\mu^k(x, y) \mid k = 1, 2, \ldots\}$ for all $x, y \in S$. Also, define $\mu^0(x, y) = 0$ if $x \neq y$ and $\mu^0(x, x) = \mu(x, x)$ otherwise.

Definition 1.2.9 Let μ be a fuzzy relation defined on a fuzzy subset σ of a set S. Then the **compliment** μ^c of μ is defined as $\mu^c(x, y) = 1 - \mu(x, y)$ for all $x, y \in S$.

Theorem 1.2.11 *Let τ, π, ρ and ν be a fuzzy relations on a fuzzy subset σ of a set S. Then the following properties hold.*

(i) $\tau \cup \pi = \pi \cup \tau$.
(ii) $\tau \cap \pi = \pi \cup \tau$.
(iii) $(\tau^c)^c = \tau$.
(iv) $\pi \cup (\rho \cup \nu) = (\pi \cup \rho) \cup \nu$.
(v) $\pi \cap (\rho \cap \nu) = (\pi \cap \rho) \cap \nu$.
(vi) $\pi \circ (\rho \circ \nu) = (\pi \circ \rho) \circ \nu$.
(vii) $\pi \cap (\rho \cup \nu) = (\pi \cap \rho) \cup (\pi \cap \nu)$.
$(viii)$ $\pi \cup (\rho \cap \nu) = (\pi \cup \rho) \cap (\pi \cup \nu)$.
(ix) $(\tau \cup \pi)^c = \pi^c \cap \tau^c$.
(x) $(\tau \cap \pi)^c = \pi^c \cup \tau^c$.
(xi) *For every* $t \in [0, 1]$, $(\tau \cup \pi)^t = \tau^t \cup \pi^t$.
(xii) *For every* $t \in [0, 1]$, $(\tau \cap \pi)^t = \tau^t \cap \pi^t$.
$(xiii)$ *If* $\tau \subseteq \rho$ *and* $\pi \subseteq \nu$, *then* $\tau \cup \pi \subseteq \rho \cup \nu$.
(xiv) *If* $\tau \subseteq \rho$ *and* $\pi \subseteq \nu$, *then* $\tau \cap \pi \subseteq \rho \cap \nu$.

Definition 1.2.12 Let μ be a fuzzy relation on σ, where σ is a fuzzy subset of a set S. Then μ is said to be **reflexive** if $\mu(x, x) = \sigma(x)$ for all $x \in S$.

When μ is a reflexive fuzzy relation on σ, it is not hard to see that $\mu(x, y) \leq \sigma(x) = \mu(x, x)$ and $\mu(y, x) \leq \sigma(x) = \mu(x, x)$ for all $x, y \in S$. In other words, when we express a fuzzy relation in a matrix form, the elements of any row or any column will be less than or equal to the diagonal element belonging to that row or column. Sometimes we say μ is reflexive on σ. Next we have some interesting properties of reflexive fuzzy relations.

Theorem 1.2.13 *Let μ and ν be fuzzy relations on a fuzzy subset σ of a set S. If μ is reflexive, then $\nu \subseteq \nu \circ \mu$ and $\nu \subseteq \mu \circ \nu$.*

Proof Let $x, z \in S$. Then $(\mu \circ \nu)(x, z) = \vee\{\mu(x, y) \wedge \nu(y, z) \mid y \in S\} \geq \mu(x, x) \wedge \nu(x, z) = \sigma(x) \wedge \nu(x, z)$. But $\nu(x, z) \leq \sigma(x) \wedge \sigma(z)$. Therefore, $\sigma(x) \wedge \nu(x, z) = \nu(x, z)$. Thus, $\nu \subseteq \mu \circ \nu$. Similarly, we can prove that $\nu \subseteq \nu \circ \mu$. ■

Corollary 1.2.14 *If μ is reflexive, then $\mu \subseteq \mu^2$.*

Corollary 1.2.15 *If μ is reflexive, then $\mu^0 \subseteq \mu \subseteq \mu^2 \subseteq \mu^3 \subseteq \cdots \subseteq \mu^\infty$.*